CLASS XII – MATHEMATICS INTEGRATION

MODULE – 5/6

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DISTANCE LEARNING PROGRAMME : AN INITIATIVE BY AEES, MUMBAI

PREVIOUS KNOWLEDGE

- **TRIGONOMETRIC IDENTITIES**
- **DIFFERENTIATION**
- **STANDARD INTEGRATION FORMULAS**

DEFINITE INTEGRALS

Let f(x) be a continuous function defined on closed interval $[a, b]$ then $\int_a^b f(x)dx$ is called definite integral of f(x) in the interval $[a, b]$.

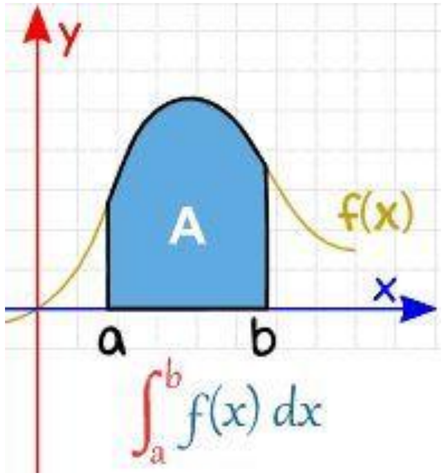
First fundamental theorem of integral calculus

f (x) be a continuous function defined on a closed interval [a, b]and (A(x) area function) A (x) = $\int_a^x f(u)du$ for all $x \in [a, b]$,then $\frac{d}{dx}A(x)$ = f(x). In other words, A(x) is an anti-derivative of f (x).

Second fundamental theorem of integral calculus

If f (x) be a continuous function defined on a closed interval [a, b] and F(x) is an anti-derivative of f (x), then,

$$\int_a^b f(x)dx = F(b) - F(a).$$



A definite integral is denoted by $\int_a^b f(x)dx$, where *a* is called the lower limit of the integral and *b* is called the upper limit of the integral. The definite integral has a unique value.

$\int_a^b f(x)dx$ defined as the definite integral of f(x) from x = a to x = b denotes area bounded by f(x) x-axis and x = a and x = b.

Note: **There is no need to write integration constant C in definite integration.**

Suppose if we consider F(x) + C then,

$$\int_a^b \mathbf{f(x)} = [\mathbf{F(x) + C}]_a^b = (\mathbf{F(b) + C}) - (\mathbf{F(a) + C}) = \mathbf{F(b) + C - F(a) - C = F(b) - F(a)}$$

EXAMPLES

1. **Evaluate $\int_1^2 (4x^3 - 5x^2 + 6x + 9)dx$ =**

Solution: $I = \int (4x^3 - 5x^2 + 6x + 9)dx = 4\frac{x^4}{4} - 5\frac{x^3}{3} + 6\frac{x^2}{2} + 9x$

$I = \frac{x^4}{4} - \frac{5x^3}{3} + 3x^2 + 9x = F(x)$

Therefore, by the second fundamental theorem of integral calculus,

$\int_1^2 (4x^3 - \frac{5x^3}{3} + 6x + 9)dx = F(2) - F(1)$

$= \left((4(2)^3 - \frac{5}{3}(2^3) + 6(2) + 9\right) - \left((4\left(1\right)^3 - \frac{5}{3}(1^2\right) + 6(1) + 9\right)$

$= (32 - \frac{40}{3} + 12 + 9) - (4 - \frac{5}{3} + 6 + 9) = 33 - \frac{35}{3} = \frac{64}{3}$

Therefore, $\int_1^2 (4x^3 - 5x^2 + 6x + 9)dx = \frac{64}{3}$

2. **Evaluate** $\int_0^{\frac{\pi}{2}} \sin^2 x\, dx$

Solution: $I = \int sin^2x\; dx = \int \frac{1-cos2x}{2}\; dx = \int \frac{1}{2}\, dx - \int \frac{cos2x}{2}\, dx$

$I = \frac{x}{2} + \frac{sin2x}{2\;(2)} = \frac{x}{2} + \frac{sin2x}{4} = F(x)$

Therefore, by the second fundamental theorem of integral calculus

$\int_0^{\frac{\pi}{2}} sin^2\, x\; dx = F(b) - F(a) = F(\frac{\pi}{2}) - F(0) = \left[\frac{\frac{\pi}{2}}{2} + \frac{sin2(\frac{\pi}{2})}{4}\right] - \left[\frac{0}{2} - \frac{sin2(0)}{4}\right]$

$= \left[\frac{\pi}{4} + 0\right] - [0 - 0] = \frac{\pi}{4}$

Therefore, $\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \frac{\pi}{4}$

3. Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x\, dx$

Solution: Let I = $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x\, dx$

$\int cosecx\ dx$ = log$|cosecx - cotx|$ = F(x)

By the second fundamental theorem of integral calculus

$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x\, dx$ = F($\frac{\pi}{4}$) - F($\frac{\pi}{6}$) = log$\left|cosec\,\frac{\pi}{4} - cot\,\frac{\pi}{4}\right|$ - log$\left|cosec\,\frac{\pi}{6} - cot\,\frac{\pi}{6}\right|$

= log$\left|\sqrt{2} - 1\right|$ - log$\left|2 - \sqrt{3}\right|$ = log $\frac{|\sqrt{2}-1|}{|2-\sqrt{3}|}$

Therefore, **$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x\, dx$ = log $\frac{|\sqrt{2}-1|}{|2-\sqrt{3}|}$**

4. Evaluate $\int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right) dx$

Solution: $\int_0^1 xe^x + \sin\frac{\pi x}{4} dx$

$$= \int \left(xe^x + \sin\frac{\pi x}{4}\right) dx = x\int e^x\,dx - \int \left\{\left(\frac{d}{dx}x\right)\int e^x dx\right\} dx + \left\{\frac{-\cos\frac{\pi x}{4}}{\frac{\pi}{4}}\right\}$$

$$= xe^x - \int e^x dx - \frac{4}{\pi}\cos\frac{x}{4}$$

$$= xe^x - e^x - \frac{4}{\pi}\cos\frac{x}{4}$$

$$= F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(1) - F(0)$$

$$= \left(e - e - \frac{4}{\pi}\cos\frac{\pi}{4}\right) - \left(0 - e^0 - \frac{4}{\pi}\cos 0^o\right)$$

$$= -\frac{4}{\pi}\left(\frac{1}{\sqrt{2}}\right) - \left(-1 - \frac{4}{\pi}\right) = 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$$

Therefore, $\int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right) dx = 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$

EVALUATION OF DEFINITE INTEGRALS BY SUBSTITUTION

Consider a definite integral of the following form

$$I = \int_a^b f[g(x)]g^{/}(x)dx$$

Substitute g(x) = t, ⇒ g ‘(x) dx = dt

When x=b, t=g(b) and when x= a, t = g(a)

Therefore, $I = \int_{g(a)}^{g(b)} f(t)dt$

Integrate the new integrand with respect to the new variable.

EXAMPLES

1. **Find the value of** $\int_0^1 xe^{x^2}dx$

Solution: let I $=\int_0^1 xe^{x^2}$

put x^2 =t ⇒ 2xdx =dt

as x→1, t → 1 and as x→0, t → 0; changing the limits

$$I = \frac{1}{2}\int_0^1 e^t\,dt = \frac{1}{2}[e^t]_0^1 = \frac{1}{2}(e^1 - e^0) = \frac{1}{2}(e-1)$$

Therefore, $\int_0^1 xe^{x^2}dx = \frac{1}{2}(e-1)$

2. Find the value of $\int_1^2 \frac{5x^2}{x^2+4x+3}\,dx$

Solution: $I = \int_1^2 \frac{5x^2}{x^2+4x+3}\,dx$

Dividing $5x^2$ by $x^2 + 4x + 3$ we get

$$\frac{5x^2}{x^2+4x+3} = 5 - \frac{20x+15}{x^2+4x+3}$$

$$I = \int_1^2 \{5 - \frac{20x+15}{x^2+4x+3}\}dx$$

$I = \int_1^2 5dx - \int_1^2 \frac{20x+15}{x^2+4x+3}\,dx = [5x]_1^2 - I_1 = 5 - I_1$ where $I_1 = \int_1^2 \frac{20x+15}{x^2+4x+3}dx$

$I_1 = \int_1^2 \frac{20x+15}{x^2+4x+3}dx$

Let $20x+15 = A\frac{d}{dx}(x^2 + 4x + 3) + B$

$20x+15 = A(2x+4) + B$

Equating coefficients of x and constant we A=10 and B = -25

$20x + 15 = 10(2x+4) - 25$

$I_1 = \int_1^2 \frac{20x+15}{x^2+4x+3}dx = \int_1^2 \frac{10(2x+4)-25}{x^2+4x+3}dx = \int_1^2 \frac{20x+40}{x^2+4x+3}dx - \int_1^2 \frac{25}{x^2+4x+3}dx$

$= 10\int_1^2 \frac{2x+4}{x^2+4x+3}dx - 25\int_1^2 \frac{1}{x^2+4x+3}dx$

Let $x^2 + 4x + 3 = t$. ⇒ (2x+4) dx = dt

$$\int_1^2 \frac{2x+4}{x^2+4x+3} = \int_1^2 \frac{dt}{t} = [\log t]_1^2 = [\log(x^2+4x+3)]_1^2$$

$$= \log 15 - \log 8 \text{ .... (1)}$$

$$\int_1^2 \frac{1}{x^2+4x+3}dx = \int_1^2 \frac{1}{x^2+4x+4-1}dx = \int_1^2 \frac{1}{(x+2)^2-1}dx$$

$$= \left[\frac{1}{2}\log\left|\frac{x+2-1}{x+2+1}\right|\right]_1^2 = \left[\frac{1}{2}\log\left|\frac{x+1}{x+3}\right|\right]_1^2 = \frac{1}{2}\log\left|\frac{2+1}{2+3}\right| - \frac{1}{2}\log\left|\frac{1+1}{1+3}\right|$$

$$\int_1^2 \frac{1}{x^2+4x+3}dx = \frac{1}{2}\log\left|\frac{3}{5}\right| - \frac{1}{2}\log\left|\frac{1}{2}\right| \text{ .... (2)}$$

Substituting (1) and (2) in I_1

$$I_1 = 10(\log 15 - \log 8) - 25\left(\frac{1}{2}\log\left|\frac{3}{5}\right| - \frac{1}{2}\log\left|\frac{1}{2}\right|\right)$$

$$= 10\log 3 + 10\log 5 - 30\log 2 - -\frac{25}{2}\log 3 + \frac{25}{2}\log 5 - -\frac{25}{2}\log 2$$

$$= \frac{45}{2}\log 5 - -\frac{5}{2}\log 3 - \frac{85}{2}\log 2$$

Substituting I_1 in I

$$I = 5 - \frac{45}{2}\log 5 + -\frac{5}{2}\log 3 + \frac{85}{2}\log 2$$

Therefore, $\int_1^2 \frac{5x^2}{x^2+4x+3}dx = 5 - \frac{45}{2}\log 5 + -\frac{5}{2}\log 3 + \frac{85}{2}\log 2$

3. **Find $\int_0^1 sin^{-1} \frac{2x}{1+x^2}$ dx**

Solution: Let I = $\int_0^1 sin^{-1} \frac{2x}{1+x^2}$ dx

Put x= tanθ $\Rightarrow$dx = $sec^2\theta$ d θ

When x = 0 , θ =0 and x= 1, $\theta = \frac{\pi}{4}$

I = $\int_0^{\frac{\pi}{4}} sin^{-1}\left(\frac{2tan\theta}{1+tan^2\theta}\right) sec^2\theta$ d θ = $\int_0^{\frac{\pi}{4}} sin^{-1}(sin2\theta)\ sec^2\theta$ d θ

= $\int_0^{\frac{\pi}{4}} 2\theta\ sec^2\theta$ d θ = 2 $\int_0^{\frac{\pi}{4}} \theta\ sec^2\theta$ d θ

Taking θ as first function and sec2θ as second function and integrating by parts,

I =2 $(\theta\ tan\ \theta)\ _0^{\frac{\pi}{4}}$- $\int_0^{\frac{\pi}{4}} 1. tan\theta\ d\ \theta$ = 2 ($\frac{\pi}{4}$ tan$\frac{\pi}{4}$ -0) -2$[\log cosx]_0^{\frac{\pi}{4}}$

= $\frac{\pi}{2}$ - 2(log$\frac{1}{\sqrt{2}}$ -log1)

= $\frac{\pi}{2}$ - log$\frac{1}{2}$ = $\frac{\pi}{2}$+ log2

Therefore, **$\int_0^1 sin^{-1} \frac{2x}{1+x^2}$ dx = $\frac{\pi}{2}$ - log$\frac{1}{2}$ = $\frac{\pi}{2}$+ log2**

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